SOLUTIONS TO EXERCISES 10.4.7, 10.5.3, 10.6.15 FROM PROBLEM SET 12

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1. Ex. 10.4.7. *Compute the Fourier sine series for the given function.*
$x^2, \qquad 0 < x < \pi$

Solution. The Fourier sine series is

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\pi} = \sum_{n=1}^{\infty} b_n \sin nx,$$

where for $n \geq 1$:

$$b_n = \frac{2}{\pi}\int_0^{\pi} x^2 \sin\frac{n\pi x}{\pi}\,dx = \int_0^{\pi} \frac{2}{\pi}x^2 \sin nx\,dx.$$

Integrate by parts, where $u = \frac{2}{\pi}x^2$ and $dv = \sin nx\,dx$, so that $du = \frac{4}{\pi}x\,dx$ and $v = -\frac{1}{n}\cos nx$. Then:

$$b_n = \left[-\frac{2}{\pi n}x^2\cos nx\right]_{x=0}^{\pi} - \int_0^{\pi}\left(-\frac{4}{n\pi}x\cos nx\right)dx = -\frac{2\pi(-1)^n}{n} + \int_0^{\pi}\frac{4}{n\pi}x\cos nx\,dx.$$

Integrate by parts, where $u = \frac{4}{n\pi}x$ and $dv = \cos nx\,dx$, so that $du = \frac{4}{n\pi}\,dx$ and $v = \frac{1}{n}\sin nx$. Then:

$$b_n = -\frac{2\pi(-1)^n}{n} + \left[\frac{4}{n^2\pi}x\sin nx\right]_{x=0}^{\pi} - \int_0^{\pi}\frac{4}{n^2\pi}\sin nx\,dx = \frac{2\pi(-1)^{n+1}}{n} + 0 - \left[-\frac{4}{n^3\pi}\cos nx\right]_{x=0}^{\pi}$$

$$= \frac{2\pi(-1)^{n+1}}{n} + \frac{4(-1)^n}{n^3\pi} - \frac{4}{n^3\pi} = \frac{2\pi(-1)^{n+1}}{n} + \frac{4}{n^3\pi}\Big((-1)^n - 1\Big).$$

Therefore, the Fourier sine series is:

$$\boxed{\sum_{n=1}^{\infty}\left(\frac{2\pi(-1)^{n+1}}{n} + \frac{4}{n^3\pi}\Big((-1)^n - 1\Big)\right)\sin nx}.$$

□

2. Ex. 10.5.3. *Find a formal solution to the given initial-boundary value problem.*
$\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial t^2}, \qquad 0 < x < \pi, \qquad t > 0,$
$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi,t) = 0, \qquad t > 0,$
$u(x,0) = x, \qquad 0 < x < \pi$

Solution. In Example 1 of Section 10.5, it was shown that a formal solution to:

$$\frac{\partial u}{\partial t} = \beta\frac{\partial^2 u}{\partial t^2}, \qquad 0 < x < L, \qquad t > 0,$$
$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi,t) = 0, \qquad t > 0,$$
$$u(x,0) = f(x), \qquad 0 < x < L$$

is:

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-\beta(n\pi/L)^2 t}\cos\frac{n\pi x}{L},$$

where for each $n \geq 0$, a_n is the coeffient in the Fourier cosine series of $f(x)$, namely:

$$a_n = \frac{2}{L}\int_0^{L} f(x)\cos\frac{n\pi x}{L}\,dx.$$

In this particular problem, $\beta = 3$, $L = \pi$, and $f(x) = x$, so the coefficients in the Fourier cosine series are:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\cos\frac{0\pi x}{\pi}\,dx = \int_0^{\pi}\frac{2}{\pi}x\,dx = \left[\frac{1}{\pi}x^2\right]_{x=0}^{\pi} = \pi.$$

For $n \geq 1$,

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos\frac{n\pi x}{\pi}\,dx = \int_0^{\pi}\frac{2}{\pi}x\cos nx\,dx.$$

Integrate by parts, where $u = \frac{2}{\pi}x$ and $dv = \cos nx\,dx$, so that $du = \frac{2}{\pi}\,dx$ and $v = \frac{1}{n}\sin nx$. Then:

$$a_n = \left[\frac{2}{n\pi}x\sin nx\right]_{x=0}^{\pi} - \int_0^{\pi}\frac{2}{n\pi}\sin nx\,dx = 0 - \left[-\frac{2}{n^2\pi}\cos nx\right]_{x=0}^{\pi} = \frac{2(-1)^n}{n^2\pi} - \frac{2}{n^2\pi} = \begin{cases} 0 & \text{if } n \text{ is even} \\ -\frac{4}{n^2\pi} & \text{if } n \text{ is odd.}\end{cases}$$

The answer is therefore:

$$\boxed{\frac{\pi}{2} - \sum_{k=0}^{\infty}\frac{4}{(2k+1)^2\pi}e^{-3(2k+1)^2 t}\cos{(2k+1)x}}.$$

□

3. Ex. 10.6.15. *Find the solution to the initial value problem.*

$\dfrac{\partial^2 u}{\partial t^2} = \alpha^2\dfrac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty, \qquad t > 0,$

$u(x,0) = f(x), \qquad -\infty < x < \infty,$

$\dfrac{\partial u}{\partial t}(x,0) = g(x), \qquad -\infty < x < \infty,$

for the given functions $f(x)$ and $g(x)$.

$f(x) = x, \qquad g(x) = x.$

Solution. In Example 2 of Section 10.6, it was shown that the solution to:

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2\frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty, \qquad t > 0,$$
$$u(x,0) = f(x), \qquad -\infty < x < \infty,$$
$$\frac{\partial u}{\partial t}(x,0) = g(x), \qquad -\infty < x < \infty,$$

is

$$u(x,t) = \frac{1}{2}\big[f(x+\alpha t) + f(x-\alpha t)\big] + \frac{1}{2\alpha}\int_{x-\alpha t}^{x+\alpha t} g(s)\,ds.$$

Therefore, the answer to this problem is:

$$u(x,t) = \frac{1}{2}\big[(x+\alpha t) + (x-\alpha t)\big] + \frac{1}{2\alpha}\int_{x-\alpha t}^{x+\alpha t} s\,ds = x + \frac{1}{2\alpha}\left[\frac{s^2}{2}\right]_{s=x-\alpha t}^{x+\alpha t} = x + \frac{1}{4\alpha}\Big((x+\alpha t)^2 - (x-\alpha t)^2\Big)$$
$$= x + \frac{1}{4\alpha}\Big(\big(x^2 + 2\alpha xt + \alpha^2 t^2\big) - \big(x^2 - 2\alpha xt + \alpha^2 t^2\big)\Big) = \boxed{x + xt}.$$

□